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SMOOTHEST LOCAL INTERPOLATION FORMULAS FOR EQUALLY SPACED DATA

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L to the (r) power — C to the m-1 power

ABSTRACT

Let a moving-average interpolation formula for equally spaced data, exact for the degree r, have a basic function $L \in C^{m-1}$ of finite support with $L^{(m)}$ piecewise continuous. Such a formula is called "smoothest" when the integral of the square of $L^{(m)}$ over the support of L is smallest. If m, r, and the support of L are given, either there is no such formula or there is a unique smoothest formula, for which L is a piecewise polynomial of degree at least r and at most $\max(r, 2m - 1)$, uniquely characterized by certain conditions on the location of its knots and the jumps occurring there. A similar result is obtained if consideration is limited to formulas that preserve (i.e., do not smooth) the given data.

AMS (MOS) Subject Classifications: 65D05, 65D07

Key Words: Interpolation, Splines, Piecewise polynomials

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*A 6-page synopsis of results, without proofs, appeared in Approximation Theory III (E. W. Cheney, ed.), Academic Press, New York, 1980.

[1]The late Hubert Vaughan was General Secretary and Actuary of the Mutual Life and Citizens' Assurance Company, Ltd., Sydney, Australia.

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SIGNIFICANCE AND EXPLANATION

In local moving-average interpolation of equally spaced data, each interpolated value is calculated as a weighted average of a few given ordinates situated near the ordinate that is being approximated. The weight applied depends only on the distance between the argument of the ordinate to which the weight is applied and the argument of the interpolated value. An example is a procedure that was often used in the construction of mathematical tables before computers became available. Every fifth or tenth value was calculated from a series expansion, and the intermediate values were obtained by local Newton-Lagrange interpolation. This amounts to fitting a piecewise polynomial function. The piecewise curve so derived had "corners," but the discontinuities in the first derivative were too small to be of any importance when a smooth mathematical function was being interpolated.

In the nineteenth century British actuaries noted that the "corners" in the curve of interpolated values were often objectionable when empirical data were being interpolated, and they published a number of local interpolation formulas in which piecewise polynomials of higher degree were fitted, and the additional degrees of freedom so obtained were utilized to secure smooth junction of adjoining polynomial arcs. In 1946 Schoenberg showed that a local moving-average interpolation formula is fully characterized by a certain function of finite support having a bell-shaped graph, similar in appearance to a probability distribution, except that it usually assumes some negative values in the tails. This he called the basic function of the interpolation formula. In 1954 the present authors published a paper in which Schwartz distributions and Schoenberg's basic-function concept were used to develop a general theory of smooth-junction local interpolation formulas.

If we fix the support of the basic function, the degree (of polynomials) for which the formula is to be exact, and the order of derivatives to be used in judging smoothness, it is shown in the present paper that there is then a unique interpolation formula of the class so defined that is, in a certain sense, smoothest. Two cases are considered: that in which the curve of interpolated values is required to pass exactly through the given data points, and the more general case in which greater smoothness is obtained by dropping this requirement.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SMOOTHEST LOCAL INTERPOLATION FORMULAS FOR EQUALLY SPACED DATA*

T. N. E. Greville and Hubert Vaughan[1]

1. INTRODUCTION.

Schoenberg pointed out in 1946 [7] that a large class of local interpolation formulas for equally spaced data can be expressed in the form

$$v_x = \sum_{\nu=-\infty}^{\infty} L(x-\nu)y_\nu , \tag{1.1}$$

where y_ν denotes a given ordinate, v_x is an interpolated value, and $L(x)$ is a given function called by Schoenberg the basic function of the interpolation formula. This class includes the numerous formulas of so-called "osculatory interpolation" published by actuarial writers (for additional references see [3]). For the latter formulas L is typically a piecewise polynomial function of finite support belonging to continuity class C^1 or C^2.

Also included is what may be called moving Newton-Lagrange interpolation, often used, before computers became available, in the preparation of tables of mathematical functions. An example would be the case in which the function f is interpolated in $(\nu h, (\nu+1)h)$, ν being any integer, by means of the cubic p uniquely determined by the four conditions

$$p(x) = f(x) \qquad (x = (\nu + j)h; \quad j = -1,0,1,2) .$$

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In this example L is a continuous piecewise cubic with support in $(-2,2)$, whose first derivative is discontinuous at $x = -2,-1,0,1,2$ (see Fig. 1). These discontinuities did not give rise to any problems in interpolating smooth mathematical functions, but are undesirable if one is interpolating empirical data.

Formula (1.1) is called reproducing when L is such that $v_\nu = y_\nu$ for every integer ν, whatever may be the values of the quantities y_ν. It is clear that (1.1) is reproducing if and only if

$$L(\nu) = \delta_{0\nu} \qquad (\nu = \ldots,-1,0,1,\ldots) \;. \tag{1.2}$$

where $\delta_{0\nu}$ is a Kronecker symbol. In practice an interpolation formula that is not reproducing smoothes as well as interpolates, since each given ordinate y_ν is, in general, replaced by an adjusted value v_ν. (Whether the adjustment does, in fact, actually increase the smoothness of the data depends on a judicious choice of L; see [7].)

Figure 1 shows the graphs of three typical basic functions. Note that Karup's formula and the Newton-Lagrange central third-difference formula are reproducing, while Jenkins' "modified" third-difference formula is not. On the other hand, note that the Newtonian graph has corners, while the others do not.

Formula (1.1) is called exact for the degree r when L is such that the formula gives exact values whenever it is used to interpolate a polynomial of degree r or less. In other words, using π_r to denote the class of polynomials of degree r or less, L is such that, for every $p \in \pi_r$, $y_\nu = p(\nu)$ for all integers ν implies $v_x = p(x)$ for all real x.

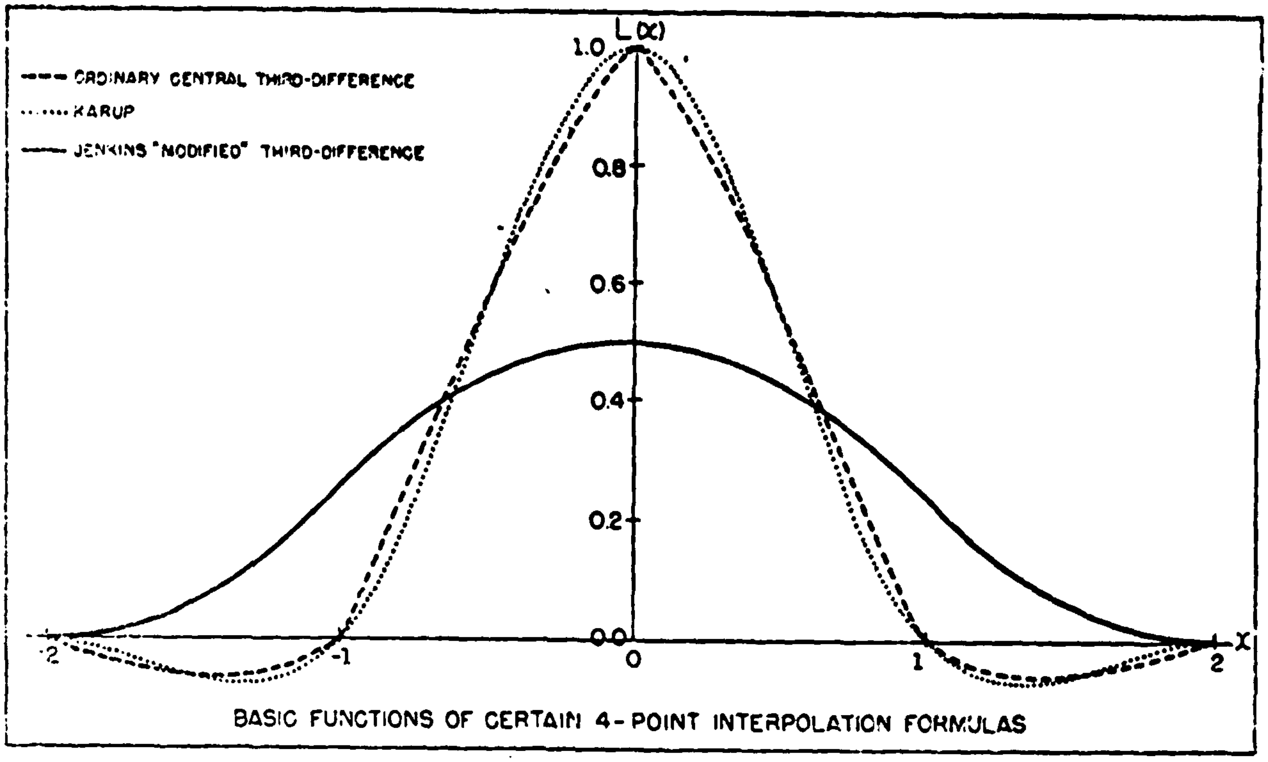

Figure 1

In the case of moving Newton-Lagrange interpolation, r is merely the degree of the polynomial arcs employed. For the actuarial formulas, r is less than the degree of the piecewise polynomial function L, the "degrees of freedom" thus gained being utilized to increase the order of continuity of L. The latter is, of course, also the order of continuity of the composite interpolating function, for if $L \in C^{m-1}$, then it is clear from (1.1) that $v \in C^{m-1}$.

When L is discontinuous (as occurs, for example, in the case of symmetrical moving Newton-Lagrange interpolation of even degree), the definition of exactness for the degree r requires interpretation. In such a case, it must be assumed that L, though discontinuous, is nevertheless such that, for every $p \in \pi_r$,

$$\phi(x) = \sum_{\nu=-\infty}^{\infty} L(x - \nu)p(\nu)$$

has only removable discontinuities, and they are removed by taking

$$\phi(x) = \phi(x + 0) = \phi(x - 0) .$$

2. MINIMIZED-DERIVATIVE FORMULAS. Let $I = (\alpha,\beta)$ be a finite open interval on the real line, and let F_{Irm} denote the set of interpolation formulas of the form (1.1) that are exact for the degree r, and have a basic function $L \in C^{m-1}$ with its support contained in I and with $L^{(m)}$ piecewise continuous. Also let F_{Irm}^{rep} denote the subset of F_{Irm} consisting of reproducing formulas. It follows from (1.2) that F_{Irm}^{rep} is empty unless $0 \in I$. By a piecewise continuous function we mean one having only jump discontinuities and at most a finite set of these.

In each of the classes F_{Irm} or F_{Irm}^{rep} we would like to find that formula which is in some sense smoothest. We shall judge smoothness by the closeness to zero of the mth derivative of the interpolating function v_x. Now, m-fold differentiation of (1.1) gives

$$v_x^{(m)} = \sum_{\nu=-\infty}^{\infty} L^{(m)}(x - \nu)y_\nu \tag{2.1}$$

almost everywhere. As we have some latitude in the choice of the basic function L, but none as regards the given ordinates y_ν, (2.1) suggests that the values of $v_x^{(m)}$ will be closer to zero than would otherwise be the case, if L is chosen so that the values of $L^{(m)}$ are, in some sense, as close to zero as possible. Accordingly, we shall call a given formula of one of the classes F_{Irm} or F_{Irm}^{rep} a <u>minimized-derivative formula</u> (mdf) of its class if the quantity

$$J = \int_{-\infty}^{\infty} [L^{(m)}(x)]^2 dx \tag{2.2}$$

assumes for the given formula its minimum value for the class in question.

The thought leading to the definition of mdf can be made more precise in the following manner. Let a be a given real number. Then, if M denotes the maximum value of $|y_\nu|$ for ν in $(a - \beta, a - \alpha + 1)$, (2.1) gives

$$|v_x^{(m)}| \leq M \sum_{\nu=-\infty}^{\infty} |L^{(m)}(x - \nu)|$$

for every x in $(a, a + 1)$. Consequently,

$$\int_0^1 |v_{a+t}^{(m)}| dt \leq M \int_\alpha^\beta |L^{(m)}(x)| dx \ . \tag{2.3}$$

If

$$\mu = \beta - \alpha$$

denotes the width of the interval I, we have, by Schwarz' inequality,

$$\int_\alpha^\beta |L^{(m)}(x)| dx \leq (\mu J)^{1/2},$$

where J is given by (2.2). Thus, (2.3) gives

$$\int_0^1 |v_{a+t}^{(m)}| dt \leq M(\mu J)^{1/2}.$$

In other words, we have shown that, for a given M, by minimizing J we minimize a certain upper bound to the integral over a unit interval of the absolute value of $v_x^{(m)}$.

Minimized-derivative interpolation formulas were previously defined by us in [3], and a few examples were given, but no general theory was developed.

In this paper we shall show that the class F_{Irm} is empty for $\mu < r + 1$, is empty or contains a single formula (of moving Newton-Lagrange interpolation) when $\mu = r + 1$, and is infinite when $\mu > r + 1$. In the latter case, we shall show that there is a unique mdf, and shall characterize this formula in a way that leads to an algorithm for its determination in any particular case.

It was previously noted that the class F_{Irm}^{rep} is empty unless the open interval I contains the origin. When this condition is satisfied, we shall find that F_{Irm}^{rep} is identical to F_{Irm} when the number of integers contained in I does not exceed $r + 1$. This implies, of course, that $r + 1 \leq \mu \leq r + 2$, but the converse is not true.

When the number of integers contained in I exceeds $r + 1$ (and 0 is among them), F_{Irm}^{rep} is a proper subset of F_{Irm}, and the former contains a unique mdf different from that associated with the latter. In this (reproducing) case too, we shall obtain a characterization of the mdf leading to an algorithm for its determination.

In order to arrive at the results just described, it is first necessary to express the requirement that (1.1) be exact for the degree r in manageable form as a set of constraints on the basic function L. A digression for this purpose is the subject of the next section.

3. MAINTENANCE OF DEGREE. Schoenberg noted in [7] that the implications of exactness of formula (1.1) for the degree r become clearer if considered in relation to a certain weaker condition. This weaker condition, in a modified form, was utilized by us in [3] and is used again here. Let H be a given function with its support contained in I, and let a function p and a function ϕ be related by the formula

$$\phi(x) = \sum_{\nu=-\infty}^{\infty} H(x - \nu)p(\nu) \ . \tag{3.1}$$

This relation may be regarded as a transformation T_H that transforms p into ϕ, or

$$\phi = T_H p \ .$$

It is evident that T_H is a linear operator. We shall say that T_H *maintains the degree* r *if it maps the space* π_r *into itself, or, in other* words, if $p \in \pi_r$ implies $\phi \in \pi_r$.

An important special case of maintenance of degree is that in which T_H annihilates π_r: in other words, ϕ is identically zero whenever $p \in \pi_r$.

Schoenberg in [7] defined a transformation T_H that preserves the degree r as one having the property that, for every $p \in \pi_r$, ϕ is a polynomial strictly of the same degree as p with the same leading coefficient. He showed that T_H has this property if its characteristic function (Fourier transform of the basic function H) has the value 1 for the argument 0 and zeros of order $r + 1$ for all nonvanishing integral multiples of 2π.

We showed in [3] that if H has an $(r + 1)$th derivative in the sense of distributions, then T_H maintains the degree r if and only if the convolution of that $(r + 1)$th derivative with every element of π_r

vanishes. A more limited result, that can be stated and proved without introducing distributions, will suffice here, and is contained in the next theorem.

If H has a piecewise continuous jth derivative, let $b_{j\zeta}$ denote the jump of $H^{(j)}$ at $x = \zeta$.

<u>Theorem 3.1</u>. If T_H maintains the degree r and if H is piecewise continuous and has piecewise continuous derivatives of all orders, then, for all real t,

$$\sum_{\nu=-\infty}^{\infty} \nu^i b_{j,t-\nu} = 0 \qquad (i = 0,1,\ldots,r;\ j = 0,1,\ldots) \ . \tag{3.2}$$

If H is a piecewise polynomial function with finite support and satisfies (3.2), then T_H maintains the degree r.

<u>Proof</u>. If we take $p(\nu) = \nu^i$ in (3.1), the left member of (3.2) is the jump of $\phi^{(j)}(x)$ at $x = t$. But, if T_H maintains the degree r and i is one of the integers $0,1,\ldots,r$, ϕ is a polynomial and this jump vanishes. Thus (3.2) is established.

On the other hand, let H be a piecewise polynomial function of finite support satisfying (3.2), and let $p \in \pi_r$ in (3.1). Then ϕ and all its derivatives are continuous everywhere. But, if d is the maximum degree of the polynomial arcs composing H, then $\phi^{(d+1)}$ vanishes almost everywhere by (3.1). Since $\phi^{(d+1)}$ is continuous everywhere, it is therefore identically zero. It follows that $\phi \in \pi_d$.

If δ denotes the "central-difference" operator defined by

$$\delta f(x) = f(x + 1/2) - f(x - 1/2) \ ,$$

we have also from (3.1)

$$\delta^{r+1}\phi(x) = \sum_{\nu=-\infty}^{\infty} p(\nu)\delta^{r+1}H(x - \nu) .$$

Expanding $\delta^{r+1}H(x - \nu)$ in terms of $H(x)$ values by the well known binomial formula and rearranging terms gives

$$\delta^{r+1}\phi(x) = \Sigma'_{\nu}H(x - \nu)\delta^{r+1}p(\nu) ,$$

where the summation Σ'_{ν} is over all the integers when r is odd, and over all the real numbers of the form integer + $1/2$ when r is even. Note that the required rearrangement of terms is permissible because the support of H is finite.

Now, since $p \in \pi_r$, $\delta^{r+1}p(\nu) = 0$ for all ν. Therefore $\delta^{r+1}\phi(x) = 0$ for all real x. But, a polynomial whose $(r + 1)$th derivative vanishes identically belongs to π_r. Therefore T_H maintains the degree r. □

If T_H maintains the degree r, then there is a differential operator of order not exceeding r, which we shall call the signature of T_H and shall denote by S_H, of the form

$$S_H = \sum_{i=0}^{r} a_i D^i , \tag{3.3}$$

that is equivalent to T_H over π_r. In other words, $S_H p = T_H p$ whenever $p \in \pi_r$. In (3.3) D denotes differentiation. The following theorem is an immediate consequence of the preceding definitions.

Theorem 3.2. T_H is exact for the degree r if and only if it maintains the degree r and its signature is the identity operator.

We can express S_H in terms of H in various ways. Thus, the coefficients a_i of (3.3) are given by

$$a_i = \frac{(-1)^i}{i!} \sum_{\nu=-\infty}^{\infty} \nu^i H(\nu) \qquad (i = 0,1,\ldots,r) \; . \tag{3.4}$$

If H is integrable, we have also

$$a_i = \frac{(-1)^i}{i!} \int_{-\infty}^{\infty} x^i H(x)\,dx \qquad (i = 0,1,\ldots,r) \; .$$

If a basic function L satisfies (1.2), then, by (3.4), the coefficients a_i in the expression (3.3) for S_L are given by

$$a_i = \delta_{0i} \qquad (i = 0,1,\ldots,r) \; .$$

Thus we have established the following corollary, previously noted in [3,7].

Corollary 3.3. The interpolation formula (1.1) is exact for the degree r if it maintains the degree r and is also reproducing.

The existence and properties of the signature S_H were established in [3] (though the term "signature" does not appear there) using the concept of distributions. However, what has been stated here is easily verified by elementary means. A similar remark applies to the following lemma, in which we take

$$\rho = \frac{1}{2}(r + 1) \; .$$

The lemma can be verified by noting (after some algebraic manipulation) that the function

$$G(x) = \sum_{j=0}^{\infty} \binom{r+j}{j} K(x - \rho - j) \tag{3.5}$$

has the required properties.

Lemma 3.4. If K is piecewise continuous with its support contained in I and T_K annihilates π_r, then there exists a piecewise continuous function G with support contained in $(\alpha + \rho, \beta - \rho)$, such that

$$\delta^{r+1}G(x) = K(x) \tag{3.6}$$

for all real x such that the left member is defined.

4. CHARACTERIZATION OF GENERAL mdf's. We note that a formula (1.1) that is exact for the degree r must satisfy

$$\sum_{\nu=-\infty}^{\infty} \nu^i L(x - \nu) = x^i \qquad (i = 0,1,\ldots,r) . \tag{4.1}$$

If the support of L is contained in I, all but a finite number of the coefficients $L(x - \nu)$ vanish automatically. If $\mu < r + 1$, it follows that there is some interval for x within which each of the $r + 1$ linearly independent functions $1,x,x^2,\ldots,x^r$ is expressible as a linear combination of r or less given functions. This is impossible. Therefore F_{Irm} is empty for $\mu < r + 1$.

If $\mu = r + 1$, then for every x such that $x - \alpha$ and $x - \beta$ are nonintegers, (4.1) can be regarded as a system of $r + 1$ linear equations in the $r + 1$ unknown values of $L(x - \nu)$. Moreover, the determinant of the matrix of coefficients of the linear system is a Vandermonde, and therefore nonvanishing. Thus, the system has a unique solution. Now, it is evident that the equations are satisfied by the $r + 1$ fundamental functions of Lagrange interpolation (or extrapolation) for the function value corresponding to the argument x, given those corresponding to the $r + 1$ arguments ν for which $L(x - \nu)$ is undetermined. Moreover, each of these fundamental functions is, indeed, a function of $x - \nu$, as (4.1) requires.

In this case of $\mu = r + 1$, L is discontinuous at those arguments $x \in I$ that differ by an integer from α or β, except in the special case in which

α and β are themselves integers and also $0 \in I$. Only in this special case is the formula reproducing and L continuous everywhere.

We conclude from the preceding discussion that the class F_{Irm} is empty for $\mu < r + 1$, and also for $\mu = r + 1$ and $m > 1$, while for $\mu = r + 1$ and $m = 0$ or 1, it is either empty or contains a single formula.

For $\mu > r + 1$, F_{Irm} contains an infinite number of formulas for every nonnegative number m, and among them, as we shall see, a unique mdf. The following theorem is the key to the characterization of this unique mdf.

Theorem 4.1. For any nonnegative integers r and m, and for $\mu > r + 1$, the class F_{Irm} contains a single formula whose basic function L satisfies the following three conditions:

(i) L is a piecewise polynomial function of degree at least r and at most $d = \max(r, 2m - 1)$.

(ii) Each knot of L is an argument that differs by an integer from α or β (or both).

(iii) The piecewise polynomial function $\delta^{r+1}L$ is given in $(\alpha + \rho, \beta - \rho)$ by a simple polynomial of degree at most $2m - 1$.

This theorem requires interpretation for $m = 0$. In that case, we interpret a polynomial of degree -1 (in condition (iii)) to mean one that is identically zero.

We shall postpone the proof of this theorem, as it will become easier after we have developed some further paraphernalia. However, without waiting to prove it, we shall proceed to demonstrate its connection with the existence of a unique mdf. For this purpose we shall need the following lemma.

Lemma 4.2. Let $K \in C^{m-1}$, with $K^{(m)}$ piecewise continuous, have its support in I, and let T_K annihilate π_r. Let H be piecewise continuous, with piecewise continuous derivatives of orders 1 to m, let T_H maintain the degree r, and let $\delta^{r+1}H$ be given in $(\alpha + \rho, \beta - \rho)$ by a simple polynomial of degree at most $2m - 1$. Then,

$$\int_{-\infty}^{\infty} H^{(m)}(x)K^{(m)}(x)\,dx = 0 \quad . \tag{4.2}$$

<u>Proof</u>. By Lemma 3.4, there exists a function $G \in C^{m-1}$, with support in $(\alpha + \rho, \beta - \rho)$, such that $G^{(m)}$ is piecewise continuous and (3.6) holds. Denoting by σ the left member of (4.2), we have

$$\sigma = \int_{-\infty}^{\infty} H^{(m)}(x)\delta^{r+1}G^{(m)}(x)\,dx \quad .$$

If $\delta^{r+1}G^{(m)}(x)$ is expanded in terms of $G^{(m)}(x)$ values, the finite support of $G^{(m)}$ then permits rearrangement of terms, so that

$$\sigma = (-1)^{r+1} \int_{-\infty}^{\infty} G^{(m)}(x)\delta^{r+1}H^{(m)}(x)\,dx \quad .$$

Since $G^{(m)}$ vanishes outside of $(\alpha + \rho, \beta - \rho)$, and $\delta^{r+1}H^{(m)}$ is given in that interval by a polynomial of π_{m-1}, say q, we have

$$\sigma = (-1)^{r+1} \int_{\alpha+\rho}^{\beta-\rho} G^{(m)}(x)q(x)\,dx \quad . \tag{4.3}$$

As $G \in C^{m-1}$, m-fold integration by parts now gives $\sigma = 0$, as required. □

<u>Theorem 4.3</u>. The unique interpolation formula determined by Theorem 4.1 is the unique mdf of the class F_{Irm}.

<u>Proof</u>. Let L be the basic function of the unique formula determined by Theorem 4.1, and L_1 the basic function of any formula of F_{Irm}. Also let J and J_1 denote the corresponding values of the quantity given by (2.2),

and let K_1 be defined by

$$L_1(x) = L(x) + K_1(x) . \tag{4.4}$$

Then, it is easily verified that K_1 fulfills the conditions required of K in Lemma 4.2. Similarly, L fulfills the requirements for H in that lemma. Therefore, by Lemma 4.2,

$$\int_{-\infty}^{\infty} L^{(m)}(x)K_1^{(m)}(x)\,dx = 0 . \tag{4.5}$$

From (4.4) and (4.5) we have

$$J_1 = J + \int_{-\infty}^{\infty} [K_1^{(m)}(x)]^2 dx .$$

It follows that $J \le J_1$. Moreover, equality holds only if $K_1^{(m)}$ vanishes almost everywhere. Therefore, in this case, $K_1^{(m-1)}$ is a step function. But, since $K_1 \in C^{m-1}$, $K_1^{(m-1)}$ is continuous, and therefore $K_1^{(m)}$ is identically zero. It follows that $K_1 \in \pi_{m-1}$. But a polynomial with finite support is identically zero, and so $L_1 = L$. □

5. CHARACTERIZATION OF REPRODUCING mdf's. We shall first dispose of the case in which F_{Irm} and F_{Irm}^{rep} are identical. Let t denote the number of integers contained in I.

Theorem 5.1. If F_{Irm} is nonempty, it is identical to F_{Irm}^{rep} if and only if $0 \in I$ and $t \le r + 1$.

Proof. We have seen that F_{Irm} is empty for $\mu < r + 1$. Therefore we must have $\mu \ge r + 1$. This implies that $t \ge r$. In fact, $t = r$ occurs only when $\mu = r + 1$ and α and β are integers. By (3.4) and Theorem 3.2, the $r + 1$ relations

$$\sum_{\nu=-\infty}^{\infty} \nu^i L(\nu) = \delta_{0i} \qquad (i = 0,1,\ldots,r) \tag{5.1}$$

must be satisfied. These may be regarded as a system of linear equations in the quantities $L(\nu)$ for the t integers ν contained in I. If $t = r + 1$, the matrix of coefficients is square. (In the special case of $t = r$, one of the end values, α or β, may be included.) The matrix is also nonsingular, because its determinant is a Vandermonde and therefore nonvanishing. Thus, the linear system has a unique solution. However, if $0 \in I$, it is evident that the values given by (1.2) are a solution, and therefore the unique solution. Hence the formula is reproducing and F_{Irm} and F_{Irm}^{rep} are identical.

As previously pointed out, there is no reproducing formula if $0 \notin I$. Now, let I be such that $t > r + 1$, and let f_1 be a formula of F_{Irm}. It follows from Theorem 4.1 that such a formula exists. If f_1 is nonreproducing, the theorem is established. Otherwise, let L_1 be the basic function of f_1 and τ the largest integer in I. Then, for some ϵ in $(0,1/2)$, $[\tau - r - 1 - \epsilon, \tau + \epsilon] \subset I$. Now, consider the interpolation formula f_2 under which, for x in $(\lambda - \epsilon, \lambda + \epsilon)$ for every integer λ,

$$v_x = \sum_{\nu=-\infty}^{\infty} L_1(x - \nu)y_\nu + k(x - \lambda + \epsilon)^m(x - \lambda - \epsilon)^m\Delta^{r+1}y_{\lambda-\tau}\ , \tag{5.2}$$

k being arbitrary, while, for all other values of x, v_x is given by the summation term only. Here Δ is the usual finite-difference operator. Evidently f_2 belongs to F_{Irm}. However, (5.2) gives

$$v_\lambda = y_\lambda + k(-1)^m \epsilon^{2m}\Delta^{r+1}y_{\lambda-\tau}\ ,$$

and the formula is clearly nonreproducing for $k \neq 0$. □

The analogue of Theorem 4.1 for the reproducing case is the following theorem.

Theorem 5.2. For any nonnegative integer r and positive integer m, and any finite interval $I = (\alpha,\beta)$ containing 0 and such that $\mu > r + 1$, the class F_{Irm}^{rep} contains a single formula satisfying the following three conditions:

(i) L is a piecewise polynomial function of degree at least r and at most $d = \max(r,2m - 1)$.

(ii) Each knot of L is either an integer or an argument that differs by an integer from α or β (or both).

(iii) The piecewise polynomial function $\delta^{r+1}L$ is given in $(\alpha + \rho, \beta - \rho)$ by a spline function of degree $2m - 1$ with simple knots. The knots of $\delta^{r+1}L$ in $(\alpha + \rho, \beta - \rho)$ are at the integers when r is odd, and at the arguments of the form integer $+ 1/2$ when r is even.

As in the case of Theorem 4.1, we shall postpone the proof of this theorem, but we shall now show its relationship to reproducing mdf's, for which the following lemma will be needed.

Lemma 5.3. Let functions K and H satisfy the same hypotheses as in Lemma 4.2 except that (i) K vanishes at the integers, and (ii) $\delta^{r+1}H$ is given in $(\alpha + \rho, \beta - \rho)$ by a spline function of degree $2m - 1$ with knots as specified in condition (iii) of Theorem 5.2. Then (4.2) holds.

Proof. The proof is the same as that of Lemma 4.2 down to equation (4.3) except that q is a spline of degree $m - 1$ with knots as specified in condition (ii), and it follows from the expression (3.5) for G that it vanishes at the knots of q. Thus m-fold integration of (4.3) gives

$$\sigma = (-1)^{m+r+1}\Sigma_x G(x)[q^{(m-1)}(x + 0) - q^{(m-1)}(x - 0)] ,$$

where the summation is over the knots of q in $(\alpha + \rho, \beta - \rho)$. Since G vanishes at these knots, $\sigma = 0$, as required. □

Theorem 5.4. The unique interpolation formula determined by Theorem 5.2 is the unique mdf of the class F_{Irm}^{rep}.

Proof. The proof is identical to that of Theorem 4.3, except that the role of Lemma 4.2 there is now assumed by Lemma 5.3. □

6. COMPACT EXPRESSION FOR mdf BASIC FUNCTIONS.

If there is a formula of F_{Irm} with a basic function that satisfies conditions (i) and (ii) of Theorem 4.1, this basic function can be regarded as a spline function of degree d with multiple knots of multiplicity $d - m + 1$. In general, therefore, it has a unique expression (see [2]) of the form

$$L(x) = \sum_{i=m}^{d} \sum_{j=0}^{n} \left[c_{ij}(x - \alpha - j)_+^i + g_{ij}(x - \beta + j)_+^i\right], \tag{6.1}$$

where n denotes the largest integer contained in μ and $y_+^i = \max(y^i, 0)$. The coefficients c_{ij} and g_{ij} are subject to the constraints arising from Theorem 3.1, which can be written as

$$\begin{array}{ll} \text{(a)} & \sum_{j=0}^{n} j^k c_{ij} = 0 \\ & \\ \text{(b)} & \sum_{j=0}^{n} j^k g_{ij} = 0 \end{array} \qquad (k = 0,1,\ldots,r;\ i = m, m+1,\ldots,d)\ . \tag{6.2}$$

If μ is an integer, the second term of the summand in (6.1) is absent (as are, of course, the constraints (6.2)(b)).

For a formula of F_{Irm}^{rep} that satisfies conditions (i) and (ii) of Theorem 5.2, in general there must be added to the right member of (6.1) the expression

$$\sum_{j\in E} h_j (x - j)_+^{2m-1} , \tag{6.3}$$

where E denotes the set of integers contained in I, and the constraints

$$\sum_{j\in E} j^k h_j = 0 \qquad (k = 0,1,\ldots,r) \tag{6.4}$$

must be satisfied. However, when α or β is an integer, the addition of (6.3) is not required.

We shall now show that by taking into account condition (iii) of Theorems 4.1 and 5.2 and by introducing certain special spline functions, we can rewrite (6.1) in a form involving a much smaller number of undetermined coefficients and can also avoid the necessity of considering separately the cases in which μ, α, or β is an integer. For this purpose we shall need the following lemma.

Lemma 6.1. For every nonnegative integer r and every positive integer $n \geq r + 1$, there is a unique polynomial $p_{rn} \in \pi_r$ such that

$$\sum_{j=1}^{n} j^k p_{rn}(j) = \delta_{0k} \qquad (k = 0,1,\ldots,r) . \tag{6.5}$$

Proof. This follows easily from well known properties of orthogonal polynomials, but it is also readily seen as follows. Equations (6.5) may be regarded as a system of $r + 1$ linear equations in the $r + 1$ coefficients of p_{rn}. The latter system has a unique solution if and only if the corresponding homogeneous system has only the trivial solution. But any solution of the homogeneous system gives rise to a polynomial $P \in \pi_r$ such that

$$\sum_{j=1}^{n} P(j)q(j) = 0 \tag{6.6}$$

for all $q \in \pi_r$. In particular, one may take $q = P$, so that (6.6) becomes a sum of squares, and therefore P vanishes for $j = 1,2,\ldots,n$. Since $n > r$, P is identically zero. □

We now define the splines of degree r,

$$S_{rn}(x) = x_+^r - \sum_{j=1}^{n} p_{rn}(j)(x-j)_+^r ,$$

$$S_{rn}^*(x) = x_+^r - \sum_{j=1}^{n} p_{rn}(j)(x+j)_+^r .$$

Parenthetically, we remark that by means of (6.5) and the identity

$$y_+^r = y^r - (-1)^r(-y)_+^r$$

it is easily shown that $S_{rn}^*(x) = (-1)^{r+1}S_{rn}(-x)$. We observe also that S_{rn} has its support in $(0,n)$ and S_{rn}^* in $(-n,0)$.

Note that condition (iii) of Theorem 4.1 or 5.2 implies, in general, that certain knots that the function $\delta^{r+1}L$ would otherwise be expected to have are absent (or reduced in multiplicity in special cases of a reproducing formula). Using the notation of (6.1), this means that

$$\Delta^{r+1}c_{ij} = \Delta^{r+1}g_{ij} = 0 \qquad (j = 1,2,\ldots,N-r-1;\ i = m,m+1,\ldots,d) ,$$

where the finite differences are taken with respect to j, and N is the largest integer less than μ. (Note that N differs from n when μ is an integer.) This implies the existence, for $i = m,m+1,\ldots,d$, of a polynomial q_i such that $c_{ij} = q_i(j)$ for $j = 1,2,\ldots,N$ but not, in general, for $j = 0$. Similar remarks apply to g_{ij}. We conclude from these facts and the constraints (6.2) that

$$\sum_{j=0}^{N} \left[c_{ij}(x-\alpha-j)_+^i + g_{ij}(x-\beta+j)_+^i\right]$$

$$= c_i S_{rN}^{(r-i)}(x-\alpha) + g_i S_{rN}^{*(r-i)}(x-\beta) \tag{6.7}$$

$$(i = m, m+1, \ldots, d) ,$$

where c_i and g_i are obtained by multiplying c_{i0} and g_{i0} by appropriate constants depending only on i and r, and, in case $i > r$, a derivative of negative order denotes that particular integral of corresponding order that vanishes identically for $x < \alpha$. A little reflection will convince the reader that the right member of (6.7) is a valid substitution even when μ is an integer (and $N = \mu - 1$). However, in the case of a reproducing formula, the expression (6.3) now must always be added, even though α or β (or both) is an integer.

Accordingly, (6.1) can be expressed uniquely in the form

$$L(x) = \sum_{i=m}^{d} \left[c_i S_{rN}^{(r-i)}(x-\alpha) + g_i S_{rN}^{*(r-i)}(x-\beta)\right] . \tag{6.8}$$

A function L satisfying conditions (i)-(iii) of Theorem 4.1 that is the basic function of a formula of F_{Irm} has, therefore, a unique expression of the form (6.8). Similarly, a function L satisfying conditions (i)-(iii) of Theorem 5.2 that is the basic function of a formula of F_{Irm}^{rep} has a unique expression of the form (6.8) with (6.3) added.

The coefficients c_i and g_i (and h_j in the reproducing case) must satisfy certain constraints. These will now be described.

(a) In the reproducing case the $r + 1$ conditions (6.4) must be satisfied. When this is the case, the expression (6.3) vanishes for $x \geq \beta$.

(b) The function given by (6.8) has its support in I when $d = r$. This is also true in the reproducing case if the preceding condition (a) is fulfilled. However, when $d = 2m - 1 > r$, (6.8) gives, for $x \geq \beta$, a polynomial of degree $d - r - 1$. The vanishing of the $d - r$ coefficients of this polynomial involves $d - r$ constraints.

(c) In the general case, exactness for the degree r requires that the $r + 1$ conditions (5.1) be satisfied.

(d) In the reproducing case, (c) is replaced by conditions (1.2). This involves t effective constraints.

(e) The form of (6.8) ensures that $\delta^{r+1}L$ shall be given in $(\alpha + \rho, \beta - \rho)$ by a polynomial in the general case, and by a polynomial except for discontinuities in the $(2m - 1)$th derivative in the reproducing case. However, in the case when $d = r > 2m - 1$, condition (iii) of Theorem 4.1 or 5.2 involves also a reduction in degree from d to $2m - 1$. This constitutes $d - 2m + 1$ constraints.

7. PROOFS OF THE CHARACTERIZATION THEOREMS.

We now have the machinery needed to prove Theorems 4.1 and 5.2.

Proof of Theorem 4.1. If there is a formula of F_{Irm} whose basic function satisfies conditions (i)-(iii) of Theorem 4.1, that function has a unique expression of the form (6.8) with parameters c_i and g_i satisfying conditions (b), (c), and (e) of the preceding section, to the extent these conditions are applicable. On the other hand, if there is an expression of this form with parameters satisfying these conditions, then it is, in fact, the basic function of such a formula.

Now, (6.8) contains $2(d - m + 1)$ undetermined parameters. The numbers of constraints involved in conditions (b), (c) and (e) are, respectively, $d - r$, $r + 1$ and $d - 2m + 1$. (Note that the integers $d - r$ and $d - 2m + 1$ are not both different from zero.) The total number of constraints is $2(d - m + 1)$, the same as the number of parameters. Without spelling out the constraints in detail, it is easily verified that they are linear equations in the parameters. The parameters must therefore satisfy $2(d - m + 1)$ equations in as many unknowns. To prove Theorem 4.1 it is sufficient to show that this linear system is nonsingular.

This is the case if the corresponding homogeneous system has only the trivial solution. In fact, the only one of the constraint equations that has a number other than 0 on its right-hand side is the one obtained by taking $i = 0$ in (5.1). Thus, a function K of the form (6.8) whose parameters satisfy the homogeneous system has the property that T_K annihilates Ψ_r. This function K therefore fulfills the requirements for both K and H in Lemma 4.2. Consequently, by Lemma 4.2,

$$\int_{-\infty}^{\infty} [K^{(m)}(x)]^2 dx = 0 .$$

By the same reasoning used in the proof of Theorem 4.3, it follows that K is identically zero. Thus, the homogeneous system has only the trivial solution. $\square$

<u>Proof of Theorem 5.2</u>. The perceptive reader may have noticed that the possibility of $m = 0$, though allowed in Theorem 4.1, is excluded in Theorem 5.2. In fact, a reproducing mdf with $m = 0$ is somewhat meaningless, for the following reason. Application to this case of the criteria that we have developed would lead to a solution in which the basic function of the corresponding mdf without the reproducing requirement is modified by arbitrarily assigning at the integers the values given by (1.2), even though these are inconsistent with the values at neighboring arguments. Thus, the resulting basic function would have removable discontinuities at the integers

in I. Strictly speaking, such a function is not piecewise continuous, and therefore is not the basic function of a formula of the class F_{Ir0}.

If there is a formula of F_{Irm}^{rep} whose basic function satisfies conditions (i)-(iii) of Theorem 5.2, that function has a unique expression of the form (6.8) with (6.3) added, and the parameters c_i, g_i, and h_j satisfy conditions (a), (b), (d), and (e) of the preceding section, to the extent these conditions are applicable. On the other hand, if there is an expression of this form, with parameters satisfying these conditions, then it is, in fact, the basic function of such a formula.

Now (6.8) and (6.3) together contain $2(d - m + 1) + t$ undetermined parameters. The number of constraints involved in conditions (a), (b), (d), and (e) are respectively, $r + 1$, $d - r$, t, and $d - 2m + 1$. The total number of constraints is $2(d - m + 1) + t$, the same as the number of parameters. As in the general case, all the constraints are linear equations in the parameters, and they constitute a linear system having a square coefficient matrix.

The remainder of the proof is the same as for Theorem 4.1, except that Lemma 5.3 now assumes the role played by Lemma 4.2 in the earlier proof.

8. SOME mdf's ARE PREVIOUSLY PUBLISHED FORMULAS. In some instances the minimized-derivative formula of a class turns out to be a previously published formula. Table I lists, for the cases known to us, the class F_{Irm} or F_{Irm}^{rep} involved, the name of the originator, the publication citation, and the year of publication. Two of the papers cited contain a large number of formulas, and in these cases the particular formula is identified. In two instances in which the published formula contains an unspecified parameter, the numerical value of the parameter that yields the mdf is given in a

footnote. The entry "Both" in the fourth column means that F_{Irm} and F_{Irm}^{rep} are identical for the case involved.

TABLE I. Previously Published Formulas that are mdf's

I	r	m	Rep or Nonrep	Originator and Citation	Publication Year
(-2, 2)	1	2	Nonrep	Jenkins [5]	1927
(-2, 2)	2	2	Both	Karup [6]	1898
(-2, 2)	2	3	Both	Greville [1] (105)	1944
(-5/2, 5/2)	2	2	Nonrep	Greville [1] (67)[2]	1944
(-5/2, 5/2)	3	2	Nonrep	Greville [1] (69)	1944
(-3, 3)	3	2	Nonrep	Greville [1] (73)[3]	1944
(-3, 3)	3	3	Nonrep	Vaughan [10] "C"	1946
(-3, 3)	3	2	Rep	Henderson [4]	1906
(-3, 3)	4	2	Both	Shovelton [8]	1913
(-3, 3)	4	3	Both	Sprague [9]	1880

[2] With $a_{13} = 13/80$.

[3] With $a_{04} = -7/108$.

REFERENCES

[1] T. N. E. Greville, The general theory of osculatory interpolation, Trans. Actuar. Soc. Amer., 45 (1944), 202-265.

[2] T. N. E. Greville, Introduction to spline functions, pp. 1-35 of Theory and Applications of Spline Functions (ed. by T. N. E. Greville), Academic Press, New York, 1969.

[3] T. N. E. Greville and H. Vaughan, Polynomial interpolation in terms of symbolic operators, Trans. Soc. Actuar., 6 (1954), 413-476.

[4] R. Henderson, A practical interpolation formula, Trans. Actuar. Soc. Amer., 9 (1906), 211-224.

[5] W. A. Jenkins, Graduation based on a modification of osculatory interpolation, Trans. Actuar. Soc. Amer., 28 (1927), 198-215.

[6] J. Karup. On a new mechanical method of graduation, Trans. 2nd Internat. Actuar. Cong., C. & E. Layton, London, 1899, pp. 78-109.

[7] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions. Part A. On the problem of smoothing or graduation. A first class of analytic approximation formulae, Quart. Appl. Math., 4 (1946), 45-99.

[8] S. T. Shovelton, On the graduation of mortality tables by interpolation, J. Inst. Actuar., 47 (1913), 284-298.

[9] T. B. Sprague, Explanation ov[4] a new formula for interpolation, J. Inst. Actuar., 22 (1880), 270-285.

[10] H. Vaughan, Some notes on interpolation, J. Inst. Actuar., 72 (1946), 482-497.

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[4]Sprague was an advocate of simplified spelling and used it in this paper.

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Let a moving-average interpolation formula for equally spaced data, exact for the degree r, have a basic function $L \in C^{m-1}$ of finite support with $L^{(m)}$ piecewise continuous. Such a formula is called "smoothest" when the integral of the square of $L^{(m)}$ over the support of L is smallest. If m, r, and the support of L are given, either there is no such formula or there is a unique

20. ABSTRACT - Cont'd.

smoothest formula, for which L is a piecewise polynomial of degree at least r and at most $\max(r, 2m - 1)$, uniquely characterized by certain conditions on the location of its knots and the jumps occurring there. A similar result is obtained if consideration is limited to formulas that preserve (i.e., do not smooth) the given data.